ISSN 2229-5518

Mathematical Models for Inertial Forces Acting on a Spinning Sphere

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Abstract : Engineering industries have made wide application of gyroscopic devices that manifest the property of the maintenance within a particular space the axis of a spinning rotor. The main component of the gyroscopic devices is the spinning rotor, which design can be different and represented by the disc, cylinder, sphere, ellipsoid, toroid, and etc. forms. The known publications represent the property of numerous gyroscopic devices by a mathematical model formulated on principle of the change in the angular momentum. Nevertheless, practical tests of gyroscopic devices do not correspond to this analytical approach. A simple spinning rotor created problems that did not have solutions for the long term. Recent investigations in this area have demonstrated that the origin of gyroscope properties is more complex than represented in numerous publications. Researchers did not pay attention to the action of inertial forcers generated by the mass elements of the spinning rotor. The applied torque on gyroscopic devices generates internal resistance and precession torques based on the action of the centrifugal, common inertial, and Coriolis forces, as well as the change in the angular momentum. These internal inertial torques act simultaneously and interdependently around two axes of gyroscopic devices, and represent the fundamental principles of the gyroscope theory. This paper presents mathematical models for the internal inertial torques generated by the mass elements and center mass of the spinning sphere. These models make it possible to describe all gyroscope properties for different design of the spinning rotor and represent novelty for machine dynamics and engineering.

Keywords: gyroscope, theory, property, torque, force

1. Introduction

Starting from the time of industrial revolution, scientists paid attention on the remarkable gyroscope properties. In 1765, L. Euler first laid out the mathematical foundations for gyroscope theory in his work on the dynamics of rigid bodies. I. Newton, J-L. Lagrange, L. Poinsot, J.L.R. D'Alembert, P-S. Laplace, L. Foucault, and other brilliant scientists added new interpretations of gyroscopic effects, which are displayed in the rotor's persistence for maintaining its plane of rotation. The applied theory of gyroscopes emerged mainly in the twentieth century [1- 4]. Numerous publications have been dedicated to the gyroscopic effects and their applications in engineering [5. 6]. Gyroscopic effects and properties are relayed in many engineering calculations for rotating parts that enable the function of numerous gyroscopic devices in engineering industries [7]. All fundamental textbooks of classical mechanics have chapters that represent gyroscope theory [8, 9]. There are many publications regarding gyroscope theory as well as many approaches and mathematical solutions that describe their properties [10, 11]. All publications contain numerous assumptions, simplifications, as well as explain the gyroscope effects in terms of the conservation of kinetic energy and by the action of the change in the angular momentum of the spinning rotor [12-14]. Some researchers intuitively pointed to the action on the gyroscope of inertial forces, which also includes the manifestation of gyroscope effects [15, 16]. However, the action of inertial forces on gyroscopes is not represented by mathematical models.

Mathematical models for gyroscope properties in known publications often do not match practical applications for gyroscope devices [17, 18]. From this, researchers have spawned artificial terms such as gyroscope resistance, gyroscope couple, gyroscope effects and some other fantastical properties [19, 20]. This is the reason that gyroscope theory still attracts many researchers seeking to find true gyroscope theory [21, 22]. However, the origin of gyroscope effects is more complex than those represented in the known theories. Recent investigations of the physical principles of gyroscope motions demonstrate four inertial forces acting upon a spinning rotor generate all gyroscopic effects. New analytical approach based on action of inertial forcers are generated by the centre mass and distributed mass element of the spinning rotor, which design and form plays significant role. Research has shown that internal torques produced by mass elements and center mass of spinning rotors manifest centrifugal, common inertial, Coriolis forces, and the change in the angular momentum. These torques interrelated and occur simultaneously, and represent the fundamental principles of gyroscope theory. Centrifugal and Coriolis forces generate resistance torque and common inertial forces and the change in the angular momentum of a spinning rotor generate the precession torque [23 - 25]. New mathematical models for acting internal torques gives significant impact to gyroscope theory and enable to describe all properties for gyroscopic devices that were unsolved. Such a new analytical approach to the gyroscope theory is formulated in terms of physical principles and understandable to users. This work demonstrates application of new mathematical models for the internal inertial torques generated by the mass elements and centre mass of the spinning sphere.

2. Analysis of centrifugal forces acting on a spinning sphere

The primary component of the gyroscopic device is the spinning rotor mounted on the axle. The inertial forces of the spinning rotor are generated by its centre mass and mass element. The designs of the spinning rotor can be different and locations and actions of the centre mass and mass element depend on the form of the spinning rotor. This work considers a spinning sphere that is rotor, at about its horizontal axis *oz* with a constant angular velocity of ω in a counter clockwise direction when viewed from the tip of axis *oz* (Fig.1). The rotor's mass elements *m* are located on the sphere whose radius is (2/3)*R* of the rotor, creating their rotating planes around axes *oz*, *oy* and *ox*. In uniform circular motions around axes separately, the value of the tangential velocity of mass elements does not change. However, the velocity is a vector quantity so that its direction changes continuously, i.e. the mass elements move with acceleration. This acceleration and rotation of mass elements represent the centrifugal forces that form the rotating forces' pseudo plane, which acts strictly perpendicular to the axis *oz* of the spinning sphere. If an external torque is applied to the spinning sphere, the rotating centrifugal forces' plane is declined and resisted opposite to the action of external torque.

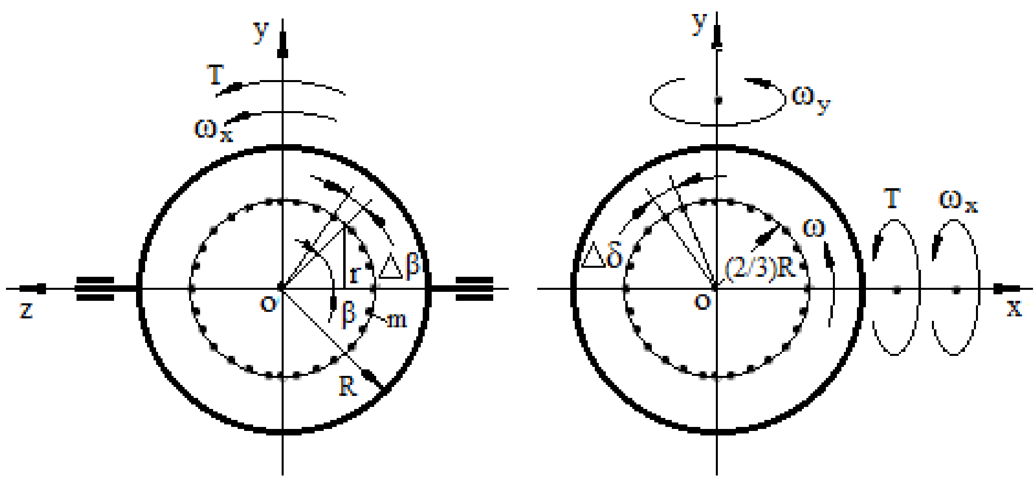
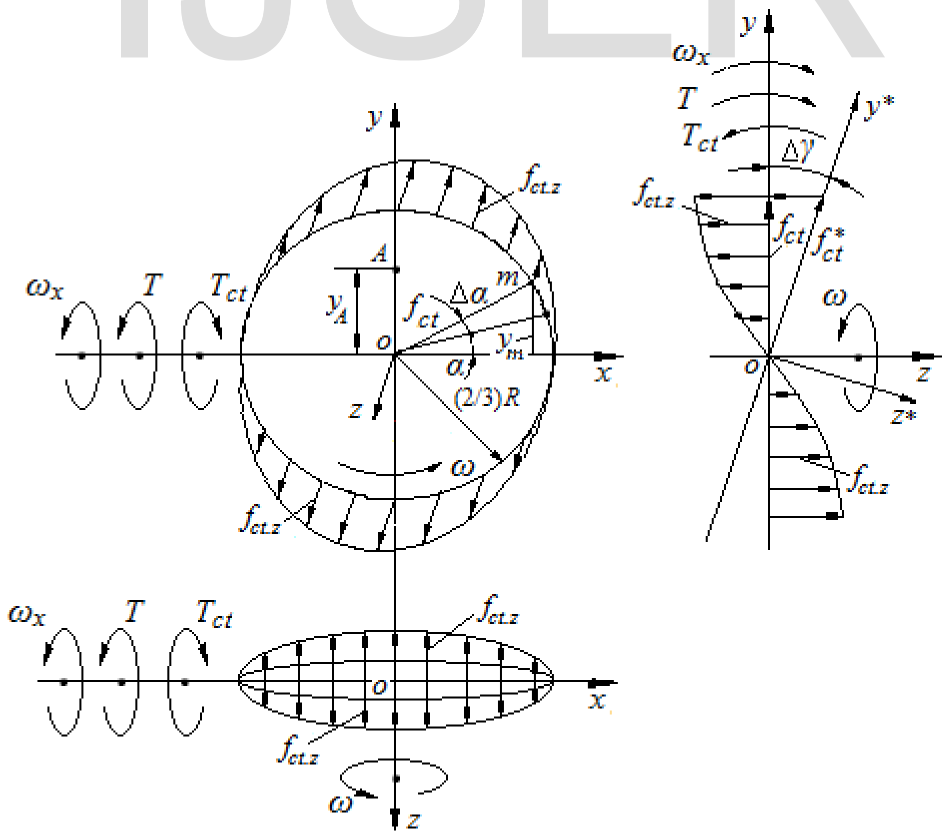


Figure 1. A scheme of the spinning sphere

Let's consider the spinning mass elements that located on the plane *xoy*. The turn of the spinning sphere's plane around axis *ox* that passes along the diameter line leads to change in the directions and locations of the centrifugal force vectors f_{ct}. The vectors f_{ct}, whose directions coincide with the line of axis *ox* (i.e. located on 0^{o} and 180^{o} from the line of axis *ox*), do not change. Other vectors of the centrifugal forces f_{ct} are located on the inclined plane and have non-identical change in their own directions. The maximal declination of vectors f^{*}_{ct} from line of axis *oy* has vectors that are located at 90^{o} and 270^{o} from the line of axis *ox* (Fig. 2). These variable directions of the centrifugal forces' vectors generate change in the vector's components $f_{ct.z}$, whose directions are parallel to the spinning rotor axle *oz*. The integrated product of components' for the vector's change in centrifugal forces $f_{ct.z}$ and their variable radius of location relative to axis *ox* generate the resistance torque T_{ct} acting opposite to the external torque. Centrifugal forces originally counteract the action of the external torque directed to change the location of the centrifugal forces' plane.



ISSN 2229-5518

Figure 2. A scheme acting forces, torques and motions of the spinning mass elements of the sphere that located on the plane *xoy*.

Similar resistance torques are generated by the mass elements located on the planes that parallel to the plane *xoy* (Fig. 1). The radius of location for other mass elements is represented by the following expression $r = (2/3)R\sin\beta$. The following analytical approach describes the resistance torque generated by the centrifugal forces of the spinning sphere as a reaction to the external torque being applied to the rotor. At this stage of mathematical development, it is possible to neglect the weight of the rotor axle and accept the bearing friction as negligible. As a starting condition, the rotating mass elements *m*, at a single instance of time, are located symmetrically in both positive and negative directions to the rotor's axis *oz*, and do not change directions instantly in space. The spinning sphere is in balance due to the rotating centrifugal forces. The action of the external torque leads to the turning of the rotating mass elements' plane *xoy* and others that parallel to one onto the small angle $\Delta\gamma$ around axis *ox* and to changing its location represented by the plane *y*ox*. The action of the external torque *T* produces a contracting moment, which manifests as a resistance torque generated by the components of change in the centrifugal forces (Fig. 2). The resistance torque produced by the centrifugal force of the mass element is expressed by the following equation:

$$\Delta T_{ct} = f_{ct.z} y_m \qquad (1)$$

where ΔT_{ct} is the torque generated by the centrifugal force of the spinning sphere's mass element; $f_{ct.z}$ is the axial component of the centrifugal force; and y_m is the distance of location in the mass element along axis *oy*.

The following expression represents the equation for the axial component of the mass element's centrifugal force (Fig. 2):

ISSN 2229-5518

$$f_{ct.z} = f_{ct}\sin\alpha\sin\Delta\gamma = mr\omega^2\sin\alpha\sin\Delta\gamma = \left[\frac{M(2/3)R\sin\alpha\sin\beta\omega^2}{4\pi}\Delta\delta\right]\sin\Delta\gamma = \frac{MR\omega^2}{6\pi}\Delta\delta\Delta\gamma\sin\alpha\sin\beta \quad (2)$$

where $f_{ct} = mr\sin\alpha\sin\beta\omega^2 = \left[\frac{M(2/3)R\omega^2\sin\alpha\sin\beta}{4\pi}\Delta\delta\right]$ is the centrifugal force of the mass element *m*; $m = \frac{M}{4\pi}\Delta\delta$, *M* is the mass of the rotor; 4π is the spherical angle of the ball; $\Delta\delta$ is the spherical angle of the mass element's location; $r = (2/3)R\sin\alpha\sin\beta$ is the radius of the mass elements location; *R* is the external radius of the sphere; ω is the constant angular velocity of the sphere; α is the angle of the mass element's location on the plane that parallel to plane *xoz*; β is the angle of the mass element's location on the plane that parallel to plane *xoy*; $\Delta\gamma$ is the angle of turn for the sphere's plane around axis *ox* ($\sin\Delta\gamma = \Delta\gamma$ for the small values of the angle).

Substituting the defined parameters into Eq. (1) yields the following equation:

$$\Delta T_{ct} = \frac{MR\omega^2}{6\pi}\Delta\delta\Delta\gamma\sin\alpha\sin\beta\times y_m = \frac{MR\omega^2}{6\pi}\Delta\delta\Delta\gamma\sin\alpha\sin\beta\times\frac{2}{3}R\sin\alpha\sin\beta = \frac{MR^2\omega^2}{9\pi}\Delta\delta\Delta\gamma\sin^2\alpha\sin^2\beta \quad (3)$$

where $y_m = (2/3)R\sin\alpha\cos\beta$ (Figs. 1 and 2) is the distance of the mass element's location on the sphere's plane relative to axis *ox*, other components are as specified above.

Equation (3) contains variable parameters whose incremental components are independent and represented by different symbols. Additionally, Eq. (3) allows for defining the integrated torque generated by the action of the centrifugal forces' axial components of the spinning sphere's mass elements, wherein all components should be presented in a form appropriate for integration. For integration, trigonometric functions are represented in differential forms. The action of the centrifugal forces' axial components represents the distributed load applied across the length of the circle and angle *α,* where the sphere's mass elements are located. Figure 2 depicts locations of the axial components of centrifugal forces $f_{ct.z}$ generated by the mass elements *m* of the spinning sphere. A distributed load can be equated with a concentrated load applied at a specific point along axis *oy*, which is centroid at the semi-circle. The location of the resultant force is the centroid (point *A*, Fig. 2) of the area under the curve, which is calculated by the known integrated Eq. (4). The distance of location of point *A* is defined by the expression y_m that represented above. Substituting Eq. (2) and other components into Eq. (4) and then transformation and simplification yields the following expression:

$$y_A = \frac{\int\limits_{\alpha=0}^{\pi}\int\limits_{\beta=0}^{\pi} f_{ct.z}\, y_m\, d\alpha d\beta}{\int\limits_{\alpha=0}^{\pi}\int\limits_{\beta=0}^{\pi} f_{ct.z}\, y_m\, d\alpha d\beta} = \frac{\int\limits_{\alpha=0}^{\pi} \frac{MR\omega^2}{6\pi}\Delta\delta\Delta\gamma \frac{2}{3} R\sin\alpha \sin\alpha d\alpha \int\limits_{0}^{\pi}\sin\beta d\beta}{\int\limits_{\alpha=0}^{\pi} \frac{MR\omega^2}{6\pi}\Delta\delta\Delta\gamma \sin\alpha d\alpha \int\limits_{0}^{\pi}\sin\beta d\beta} =$$

$$\frac{\frac{MR\omega^2}{6\pi}\Delta\delta\Delta\gamma \int\limits_{\alpha=0}^{\pi}\frac{2}{3}R\sin^2\alpha d\alpha}{\frac{MR\omega^2}{6\pi}\Delta\delta\Delta\gamma \int\limits_{\alpha=0}^{\pi}\sin\alpha d\alpha} = \frac{R\int\limits_{0}^{\pi}(1-\cos 2\alpha)d\alpha}{3\int\limits_{0}^{\pi}\sin\alpha d\alpha} \qquad (4)$$

where the expression $\frac{MR\omega^2}{6\pi}\Delta\delta\Delta\gamma$ is accepted at this stage of computing as constant for the Eq. (4); the expression $\sin^2\alpha = (1-\cos 2\alpha)/2$ is a trigonometric identity that replaced in the equation, and other parameters are as specified above.

Equation (3) and its components are expressed for integration by a differential form where $\sin^2\alpha = \int\limits_0^{\pi} 2\sin\alpha\cos\alpha d\alpha$ and $\sin^2\beta = \int\limits_0^{\pi} 2\sin\beta\cos\beta d\beta$; Substituting defined components into Eq. (3), and presenting by the integral form. The solution of integral is considered for the semi-circle. The upper limit for $d\delta$ is represented by the area of hemisphere. Substituting defined parameters and expressing by the integral forms, the following equation emerges:

$$\int_0^{T_{ct}} dT_{ct} = \frac{MR\omega^2}{9\pi}\int_0^{2\pi} d\delta \int_0^{\gamma} d\gamma \times \int_0^{\pi} 2\sin\alpha\cos\alpha d\alpha \int_0^{\pi} 2\sin\beta\cos\beta \times \frac{R\int_0^{\pi}(1-\cos 2\alpha)d\alpha}{3\int_0^{\pi}\sin\alpha d\alpha}$$

Solving integral Eq. (5) yields the following result:

$$T_{ct}\Big|_0^{T_{ct}} = \frac{MR\omega^2}{9\pi}\left(\delta\Big|_0^{2\pi}\right)\left(\gamma\Big|_0^{\gamma}\right)\left(2\sin^2\alpha\Big|_0^{\pi/2}\right)\left(2\sin^2\beta\Big|_0^{\pi/2}\right)\frac{R\left(\alpha-\frac{1}{2}\sin 2\alpha\right)\Big|_0^{\pi}}{-3\cos\alpha\Big|_0^{\pi}}$$

The change of the upper limit for the trigonometric expression sinus leads to increasing the result twice. Solution of the expression giving rise the following

$$T_{ct} = \frac{MR\omega^2}{9\pi}2\pi(\gamma-0)(2-0)(2-0)\frac{R(\pi-0)}{-3(-1-1)} = \frac{4MR^2\omega^2\pi}{27}\gamma \qquad (6)$$

where all parameters are as specified above.

The rate change in the torque T_{ct} per time is represented by the following differential equation:

$$\frac{dT_{ct}}{dt} = \frac{4MR^2\omega^2\pi}{27}\frac{d\gamma}{dt} \qquad (7)$$

where $t = \alpha/\omega$ is the time taken relative to the angular velocity of the spinning sphere, and other parameters are as expressed above

Then, the differential of time is: $dt = \frac{d\alpha}{\omega}$; while the expression $\frac{d\gamma}{dt} = \omega_x$, is the angular velocity of the spinning sphere's precession around axis *ox*. Substituting the defined components into Eq. (7) and transformation yield a new differential equation with the following expression:

$$\frac{\omega dT_{ct}}{d\alpha} = \frac{4MR^2\omega^2\omega_x\pi}{27} \tag{8}$$

Separating the variables of Eq. (8) and transformation yield the following equation:

$$dT_{ct} = \frac{4MR^2\omega\omega_x\pi}{27}d\alpha\ . \tag{9}$$

Equation (9) is represented by the integral form at defined limits and yields the following integral

equation:

$$\int_0^{T_{ct}} dT_{ct} = \int_0^{\pi} \frac{4MR^2\omega\omega_x\pi}{27}d\alpha \tag{10}$$

Solving Eq. (10) yields the following result:

$$T_{ct}\Big|_0^{T_{ct}} = \frac{4MR^2\omega\omega_x\pi}{27}\alpha\Big|_0^{\pi} \tag{11}$$

When the centrifugal forces act on the upper and lower sides of the sphere's plane, then the total resistance torque T_{ct} is obtained when the result of Eq. (11) is increased twice:

$$T_{ct} = \frac{2\times 4\pi^2 MR^2\omega\omega_x}{27} = \frac{5\times 2\times 2\times 2\pi^2 MR^2\omega\omega_x}{27\times 5} = \frac{20}{27}\pi^2 J\omega\omega_x \tag{12}$$

where $J = 2MR^2/5$ is the sphere mass moment of inertia, other parameters are as specified above.

Analysis of Eq. (12) shows that the resistance torque generated by the centrifugal forces of the spinning sphere depends proportionally on its mass moment of inertia and angular velocity, as well as the angular velocity of the precession. Absence of an external torque means that the angular velocity of the forced precession $\omega_x = 0$. Then, Eq. (12) gives the resistance torque's equation of the centrifugal forces $T_{ct} = 0$, which is a natural result. The action of the resistance torque generated by the centrifugal forces of the mass elements is only manifested in the case of action by the external load torque T. The direction of the resistance torque's action and the direction of the angular velocity of precession are opposite to each other. It means that the resistance torque generated by the centrifugal forces is the restraining torque.

3. Analysis of inertial forces acting on a spinning sphere

The uniform circular motion of the spinning sphere experiences the tangential velocity of its mass elements. In the case of acting on an external torque that is applied to a spinning sphere, as considered in section 1, the plane of the spinning sphere turns around axis *ox* (Fig. 2). This turn leads to a change in the direction of the mass elements' tangential velocity and produces the acceleration and inertial forces of the rotating mass elements. The turn in the plane of the spinning sphere around axis *ox* leads to non-identical change in the directions of the tangential velocity vectors. The maximal changes in direction have the velocity vectors *V** of the mass element located on the line of axis *ox* (Fig. 3). The two vectors *V* do not have any changes, whose direction is parallel to the line of axis *ox*, i.e. located on 90^{o} and 270^{o} from the line of axis *ox*. These variable directions in the tangential velocity vectors generate change in the vector's components V_z whose directions are parallel to the spinning rotor's axle *oz*.

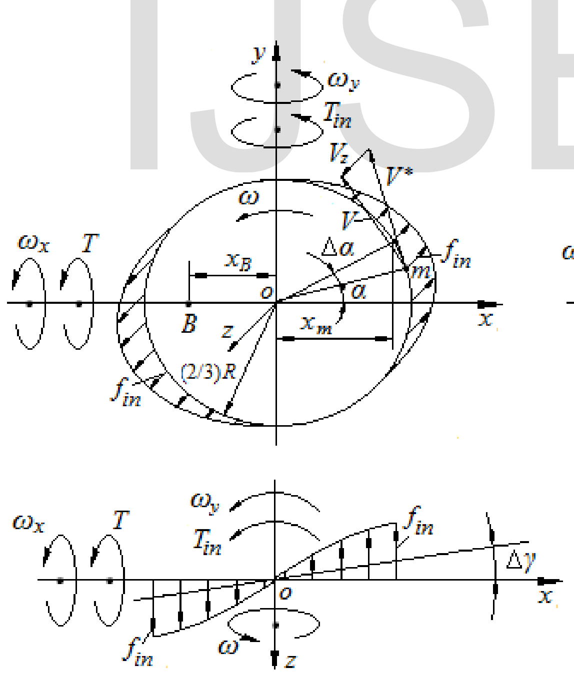


Figure 3. A scheme of acting forces, torques and motions of the spinning sphere

The change in the velocity vectors refers to the accelerated motions of the spinning sphere's mass elements that generate their inertial forces. The integrated product of components for the vector change in inertial forces f_{in} of the mass elements and their variable radius of location relative to the turn axis *oy* generate the inertial torque T_{in} acting around axis *oy*. Inertial torque is originally the result of the external torque's action and represents the precession torque that is directed to changing the location of the spinning sphere's plane around axis *oy*.

The precession torque generated by the inertial force of the mass element is expressed by the following equation:

$$\Delta T_{in} = f_{in} x_m = m a_z x_m \tag{13}$$

where ΔT_{in} is the torque generated by the inertial force of the spinning sphere's mass element f_{in}; a_z is the acceleration of the mass element *m* along axis *oz*; and x_m is the distance to the mass element's location along axis *ox*. Other components are represented in section 1.

The expression for the mass element *m* is represented by the component of Eq. (2) in section 1. The expression for the distance x_m for the mass element's location along axis *ox* is represented by Eq. (4), but with change to the indices of axes and forces. The equation for the acceleration a_z of the mass element is defined by the first derivative of change in the tangential velocity, whose value depends on the angle of its location on the spinning sphere, which is variable with time. The expression for a_z is presented by the following equation:

$$\alpha_z = \frac{dV_z}{dt} = \frac{d[V\cos\alpha(t)\sin\Delta\gamma]}{dt} = V\Delta\gamma\sin\alpha\frac{d\alpha}{dt} = \frac{2}{3}R\omega^2\Delta\gamma\sin\alpha\sin\beta \tag{14}$$

where $a_z = dV_z/dt$ is the acceleration of the mass element along axis *oz*; $V_z = V\cos\alpha(t)\sin\Delta\gamma$ is the change in the tangential velocity *V* of the mass element; $\Delta\gamma$ is the angle of the turn of the spinning sphere's plane around axis *oy* ($\sin\Delta\gamma = \Delta\gamma$ for the small values of the angle); $V =$

$(2/3)R\omega\sin\alpha\sin\beta$, where R is the radius of the sphere; $\omega = d\alpha/dt$ is the angular velocity of the spinning sphere; α is the angular location of the mass element; t is the time.

Substituting the defined parameters and Eq. (2) into Eq. (1) yields the following equation:

$$\Delta T_{in} = \frac{M}{4\pi}\times\Delta\delta\times\Delta\gamma\times\frac{2}{3}R\omega^2\sin\alpha\sin\beta\times x_m \tag{15}$$

where all parameters are as specified above.

The following solution is the same as for Eq. (3) of section 1, and yields the equation for the precession torque whose expression is as follows:

$$T_{in} = \frac{20}{27}\pi^2 J\omega\omega_x \tag{16}$$

Equation (16) represents the precession torque generated by the mass elements of the spinning sphere. The external torque, as applied to the spinning sphere and acting around axis *ox*, generates and directs its angular velocity of precession around axis *oy* into a counter clockwise direction. The precession torque generated by the inertial forces and resistance torque generated by the centrifugal forces are both originated by one external load torque. These two torques are expressed by a single equation albeit acting around different axes. Blocking the rotor's motion around axis *oy* leads to deactivation of the resistance torque produced by the centrifugal forces acting around axis *ox* and vice versa. Separate action by each torque is impossible and represents a new gyroscopic property. The change in the angular momentum represents a precession torque generated by the center mass of the spinning sphere and expressed by the well-known equation

$T_{am} = J\omega\omega_x$ where all parameters are as specified above. The total precession torque acting around axis *oy* is represented a sum of the precession torques generated by the inertial forces of the mass elements and the change in the angular momentum whose equation is as follows:

$$T_p = T_{in} + T_{am} = \left[\frac{20}{27}\pi^2 + 1\right] J\omega\omega_x \tag{17}$$

where T_p is the total precession torque acting around axis *oy*. Other components are as specified above.

4. Analysis of Coriolis forces acting on a spinning sphere

In classical mechanics, Coriolis acceleration and force are a product of the linear motion of a mass on a rotating disc. The action of the Coriolis acceleration and force is revealed in the spinning sphere under the external torque. The resulting action of Coriolis force, generated by the mass elements of the spinning sphere, is expressed as the integrated resistance torque acting opposite to the action of the external torque. Figure 4 depicts the mass element *m* that travels in circle on the sphere's plane, which turns on the plane *yoz* in the precession angle $\Delta\gamma$ around axis *ox*. This turn leads to change in the direction of the tangential velocity of mass elements, and produces the acceleration and Coriolis forces of the rotating mass elements. The turn of the spinning sphere's plane around axis *ox* leads to a non-identical change in the directions of the tangential velocity vectors. The maximal changes in direction have the velocity vectors V^* of the mass element located on the line of axis *ox* (Fig. 4). The two vectors *V* do not have any changes, whose directions are parallel to the line of axis *ox*, i.e. located on 90^o and 270^o from the line of axis *ox*. These variable directions of the tangential velocity vectors generate the change in the vector's components V_z whose directions are parallel to the spinning sphere's axle *oz*. Changes in the value of the velocity are expressed as an acceleration of the mass elements and their inertial forces.

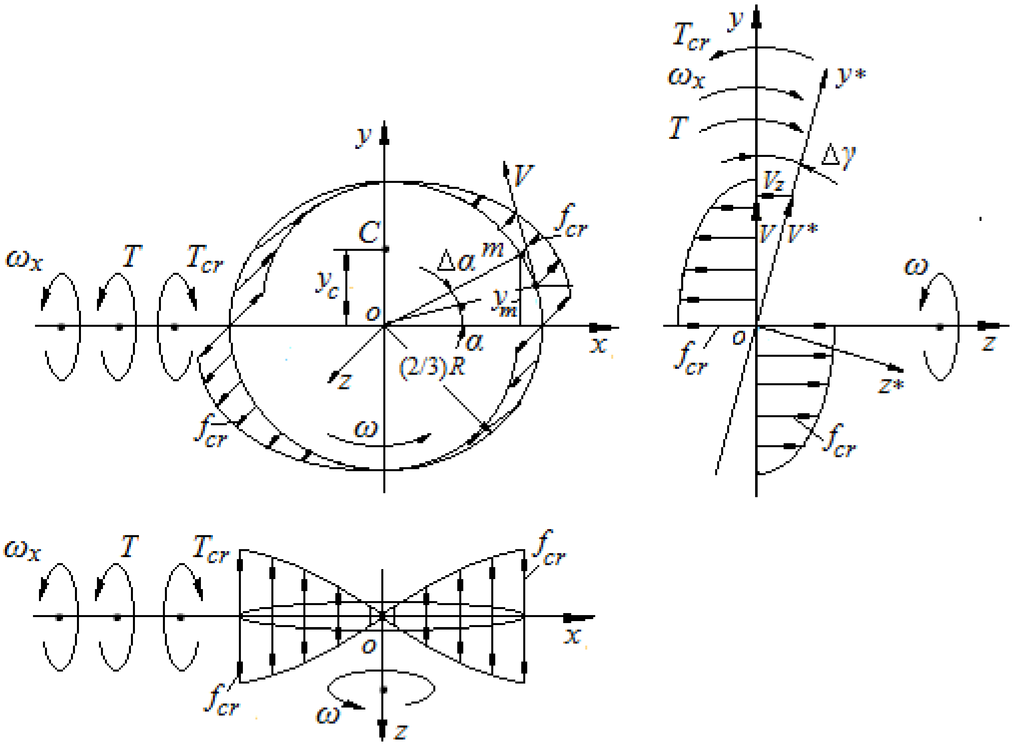


Figure 4. A scheme acting forces, torques and motions of the spinning sphere

The resistance torque generated by the Coriolis force of the mass elements is expressed by the following equation:

$$\Delta T_{cr} = f_{cr} y_m = m a_z y_m \tag{18}$$

where ΔT_{cr} is the torque generated by Coriolis force f_{cr} of the spinning sphere's mass element m; a_z is the acceleration of the mass element along axis *oz*; and $y_m = (2/3)R\sin\alpha\sin\beta$ is the distance to the mass element's location along axis *oy;* other components are represented in Eq. (2) of section 1.

The expression for mass element m is represented by the component of Eq. (2) in section 1. The equation for Coriolis acceleration a_z of the mass element is defined by the first derivative of the change in the tangential velocity, whose value depends on the angle of its location on the

ISSN 2229-5518

plane *yoz*, which is variable with time [23]. The expression for a_z is represented by the following equation:

$$\alpha_z = \frac{dV_z}{dt} = \frac{d[V\cos\alpha\sin\Delta\gamma(t)]}{dt} = V\cos\alpha\frac{d(\Delta\gamma)}{dt} = \frac{2}{3}R\omega\omega_x\cos^2\alpha\sin\beta \qquad (19)$$

where $a_z = dV_z/dt$ is the Coriolis acceleration of the mass element along axis *oz*; $V_z = V\cos\alpha\sin\Delta\gamma(t)$ is the change in the tangential velocity V of the mass element; $\Delta\gamma(t)$ is the angle of turn in the spinning sphere's plane around axis *ox* ($\sin\Delta\gamma(t) = \Delta\gamma(t)$ for the small values of the angle); $V = (2/3)R\omega\cos\alpha\sin\beta$, where R is the radius of the sphere; α and β is the angular location of the mass element (section 2); $\omega_x = \Delta\gamma(t)/dt$ is the angular velocity of precession around axis *ox*; t is the time. Then Coriolis force is represented by the following expression:

$$f_{cr} = \frac{M\Delta\delta}{4\pi}\frac{2}{3}R\omega\omega_x\cos^2\alpha\sin\beta = \frac{MR\Delta\delta}{6\pi}\omega\omega_x\cos^2\alpha\sin\beta \qquad (20)$$

Substituting these defined parameters into Eq. (18) and transformation yields the following equation:

$$\Delta T_{cr} = \frac{MR\omega\omega_x\Delta\delta}{6\pi}\cos^2\alpha\sin\beta\times\frac{2}{3}R\sin\alpha\sin\beta \qquad (21)$$

The Coriolis forces represent the distributed load applied along the length of the circle where the sphere's mass elements are located. Figure 4 depicts the locations of Coriolis forces generated by the motion of the spinning sphere's mass elements *m* around axes *oz* and *ox*. A distributed load can be equated with a concentrated load applied at a specific point along the semi-circle. The location of the resultant force is the centroid (point *C*, Fig. 4) of the area under

ISSN 2229-5518

the Coriolis force's curve calculated by Eq. (4), but with its own symbols. Substituting the defined parameters into Eq. (4) and transformation yields the following equation:

$$y_C = \frac{\int\limits_{\alpha=0}^{\pi}\int\limits_{\beta=0}^{\pi} f_{cr} y_m d\alpha d\beta}{\int\limits_{\alpha=0}^{\pi}\int\limits_{\beta=0}^{\pi} f_{cr} d\alpha d\beta} = \frac{\int\limits_{\alpha=0}^{\pi} \frac{MR\omega\omega_x\Delta\delta}{6\pi}\frac{2}{3}R\cos^2\alpha\sin\alpha d\alpha\int\limits_0^{\pi}\sin^2\beta d\beta}{\int\limits_{\alpha=0}^{\pi}\frac{MR\omega\omega_x\Delta\delta}{6\pi}\cos^2\alpha d\alpha\int\limits_0^{\pi}\sin^2\beta d\beta} =$$
$$\frac{\frac{MR\omega\omega_x\Delta\delta}{6\pi}\int\limits_{\alpha=0}^{\pi}\frac{2}{3}\cos^2\alpha d\cos\alpha\int\limits_0^{\pi}\sin^2\beta d\beta}{\frac{MR\omega\omega_x\Delta\delta}{6\pi}\int\limits_{\alpha=0}^{\pi}\cos^2\alpha d\alpha\int\limits_0^{\pi}\sin^2\beta d\beta} = \frac{\frac{2}{3}R\int\limits_0^{\pi}\cos^2\alpha d\cos\alpha}{\frac{1}{2}\int\limits_0^{\pi}(1+\cos 2\alpha)\alpha d\alpha} \tag{22}$$

where the expression $\frac{MR\omega\omega_x\Delta\delta}{6\pi}$ is accepted as constant for the Eq. (21), the expression $\cos^2\alpha = (1/2)(1+\cos 2\alpha)$ is a trigonometric identity that replaced in the equation and other parameters are as specified above.

Equation (21) is expressed by a differential form, when substituting Eq. (22). Then replacing $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha) = \frac{1}{2}\int\limits_0^{\pi}\left(1-\frac{\sin 2\alpha}{2}\right)d\alpha$, $\sin\beta = \int\limits_0^{\pi}\cos\beta d\beta$ and the expression y_m by y_C in forms of the integral expression with change of defined limits, the following equation emerges:

$$\int\limits_0^{T_{cr}} dT_{cr} = \frac{MR\omega\omega_x}{6\pi}\times\int\limits_0^{2\pi}d\delta\times\frac{1}{2}\int\limits_0^{\pi}\left(1-\frac{\sin 2\alpha}{2}\right)d\alpha\int\limits_0^{\pi}\cos\beta d\beta\times\frac{\frac{2}{3}R\int\limits_0^{\pi}\cos^2\alpha d\cos\alpha}{\frac{1}{2}\int\limits_0^{\pi}(1+\cos 2\alpha)\alpha d\alpha} \tag{23}$$

Solving integral Eq. (23) yields the following result:

$$T_{cr}\Big|_0^{T_{cr}} = \frac{MR\omega\omega_x}{6\pi}\times\left(\delta\Big|_0^{2\pi}\right)\times\frac{1}{2}\left(\alpha+\cos 2\alpha\Big|_0^{\pi}\right)\times 2\sin\beta\Big|_0^{\pi/2}\times\frac{-4R\cos^3\alpha\Big|_0^{\pi}}{9\left(\alpha+\dfrac{\sin 2\alpha}{2}\right)\Big|_0^{\pi}}$$

where the limits of integration for the trigonometric expression of sinus are taken for the quarter the circle and result is increased twice.

When the Coriolis forces act on the upper and lower sides of the sphere's plane, then the total resistance torque T_{cr} is obtained when the result of Eq. (23) is increased twice:

$$T_{cr} = 2\times\frac{MR\omega\omega_x}{6\pi}\times(2\pi-0)\times\frac{1}{2}(\pi+1-1)\times(2-0)\times\left(\frac{-4R(-1-1)}{9(\pi+0)}\right)= \frac{8MR^2\omega\omega_x}{27}=\frac{20}{27}J\omega\omega_x \quad (24)$$

where $J = 2MR^2/5$ is the sphere mass moment of inertia, other parameters are as specified above.

The analysis of Eq. (24) shows that the resistance torque generated by Coriolis forces of the spinning sphere's mass elements depends proportionally on the mass moment of the sphere's inertia, its angular velocity, and on the angular velocity of the precession. Deactivation of the external torque means that the angular velocity of the precession is $\omega_x = 0$, and hence the resistance torque generated by the Coriolis forces is also deactivated. The directions of the resistance torque and angular velocity of precession are opposite to each other. It means that the resistance torque generated by the Coriolis forces is the restraining torque. The total resistance torque acting around axis *ox* is represented a sum of the resistance torques generated by the centrifugal and Coriolis forces of the mass elements whose equation is as follows:

$$T_r = T_{ct}+T_{cr} = \left[\frac{20}{27}\pi^2+\frac{20}{27}\right]J\omega\omega_x = \frac{20}{27}(\pi^2+1)J\omega\omega_x \quad (25)$$

where T_r is the total resistance torque acting around axis *ox*. Other components are as specified above.

The total precession torque acting around axis *oy* is represented a sum of the precession torques generated by the common inertial forces and the change in the angular momentum of the sphere whose equation is as follows:

$$T_p = T_{in} + T_{am} = \left[\frac{20}{27}\pi^2 + 1\right] J\omega\omega_x \qquad (26)$$

where T_p is the total precession torque acting around axis *oy*. Other components are as specified above.

5. Working example

The sphere has a mass of 1.0 kg and a radius of 0.1 m at about the spin axis. The sphere is spinning at 3000 rpm. An external torque of 0.5 Nm acts on the sphere, which precesses with an angular velocity of 0.05 rpm. These are used to determine the value of the resistance and precession torques generated by the centrifugal, common inertial and Coriolis forces, as well as the change in the angular momentum of the spinning sphere (Fig. 1). Solving this problem is based on Eqs. (18) and (25). Substituting the initial data into the aforementioned equations and transformation yield the following result:

$$T_r = T_{ct} + T_{cr} = \frac{20}{27}(\pi^2 + 1) J\omega\omega_x = \frac{20}{27}(\pi^2 + 1)\frac{2MR^2}{5}\omega\omega_x =$$

$$\frac{20}{27}(\pi^2 + 1) \times \frac{2.0 \times 0.1^2}{5} \times \frac{3000 \times 2\pi}{60} \times \frac{0.05 \times 2\pi}{60} = 0.052977\,Nm$$

$$T_p = T_{in} + T_{am} = \left[\frac{20}{27}\pi^2 + 1\right] J\omega\omega_x = \left[\frac{20}{27}\pi^2 + 1\right]\frac{2MR^2}{5}\omega\omega_x =$$

$$\left[\frac{20}{27}\pi^2 + 1\right] \times \frac{2.0 \times 0.1^2}{5} \times \frac{3000 \times 2\pi}{60} \times \frac{0.05 \times 2\pi}{60} = 0.0546829 Nm$$

where T_r and T_p are respectively the resistance and precession torques generated by the mass elements and center mass of the spinning sphere.

6. Results and discussion

Analysis of the physical principles behind the acting forces and motions in a gyroscopic devices enables a state for actual gyroscope effects that have more complex origin than represented by known publications. New studies of the gyroscopic properties have shown that gyroscope's motions result from action of the internal torques produced by the inertial forces generated by the mass elements and center mass of the spinning rotor. The value of these torques depends on the form of the spinning rotor, which geometry can be different designs. The internal torques are generated by the centrifugal, common inertial, and Coriolis forces of the mass element, as well as the change in the angular momentum. The mathematical models for internal torques acting on the spinning sphere are derived and represented in Table 1.

Table 1. Equations of the internal torques acting in the spinning sphere.

Type of the torque generated by	Equation	Percentage of action (%)
Centrifugal forces	$T_{ct} = T_{in} = \frac{20}{27}\pi^2 J\omega\omega_x$	44.67
Inertial forces		44.67
Coriolis forces	$T_{cr} = \frac{20}{27} J\omega\omega_x$	4.52

ISSN 2229-5518

Change in an angular momentum	$T_{am} = J\omega\omega_x$	6.11
Total		100
Resistance torque $T_r = T_{ct} + T_{cr}$	$T_r = \frac{20}{27}(\pi^2 + 1)J\omega\omega_x$	49.18
Precession torque $T_p = T_{in} + T_{am}$	$T_p = \left(\frac{20}{27}\pi^2 + 1\right)J\omega\omega_x$	49.82
Total		100

These torques proportionally depend on the mass moment of inertia and angular velocity of the spinning sphere as well as on the angular velocity of its precession. The inertial forces generated by the mass elements of the spinning sphere are active physical components as the change in the angular momentum. The later one does not play the first role in gyroscope physics. The torques generated by centrifugal and common inertial forces are represented by one equation. However, the action of these torques applied to different axes of the spinning sphere that are perpendicular to the other. The torque generated by the centrifugal and Coriolis forces represents the resistance torques. The torques generated by the common inertial forces and the change in the angular momentum represents the precession torque. All torques are acting at one time and interdependently around two axes in a spinning sphere and manifest the gyroscope properties. New mathematical models enable descriptions for all gyroscope properties and are useful for modeling their behavior of the spinning sphere.

7. Conclusion

In classical mechanics, the gyroscope theory is one of the most complex and intricate in terms of analytical solutions. Known mathematical models for the theory are based on action of the load torque and the change in the angular momentum. Such models have so far failed to adequately address the numerous practical problems. In contrast, the new mathematical models for the gyroscope's torques consider interdependent action at one time of several inertial forces of the rotating mass elements and center mass of the spinning rotor. The origin of these forces is well-known in classical mechanics. These models are based on their combined action, which would manifest the gyroscope's resistance and precession torques acting along different axes. The new analytical approach is thus distinguishable from those in known publications that tend to have complex numerical modelling. The new mathematical models for the spinning sphere torques renders it possible to solve enduring problems relating to gyroscopic devices and clearly demonstrates the physical principles behind the acting forces and motions. In that vein, this is also a good example of the educational processes. The new analytical approach clearly describes gyroscope properties in a new light while setting forth new challenges for future studies of the gyroscope theory.

Notation

f_{ct}, $f_{cr.}$, $f_{in.}$ – centrifugal, Coriolis and inertial forces, respectively, generated by mass elements of a spinning sphere

J – mass moment of inertia of a sphere

M – mass of a sphere

m – mass element of a sphere

R – external radius of a sphere

T – load external torque

T_{ct}, $T_{cr.}$, $T_{in.}$ T_{am} – torque generated by centrifugal, Coriolis and inertial forces and a change in the angular momentum, respectively

t – time

y_c, y_m – centroid and distance of location of mass element along axis

$\Delta\alpha$, α – increment angle and angle of the turn for a sphere around own axis, respectively

β - angle of location the mass element of a sphere at the plane *zoy*

$\Delta\delta$ – spherical angle of the sphere's a mass element

$\Delta\gamma$ – angle of inclination of a sphere's plane

ω – angular velocity of a sphere

$\omega_{x,}$ ω_y – angular velocity of precession around axes *ox* and *oy*, respectively

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